14.3 Aufgaben zu Gleichungen

Aufgabe 1 Bringe auf Normalform

a) $\sqrt{x-a} - b = 0 \quad | +b$

$\sqrt{x-a} = b \quad | ()^2$

$\boxed{x - a = b^2} \quad | -b^2$ ✓

$x - a - b^2 = 0 \quad | \cdot(-1)$

$\underline{b^2 - x + a = 0}$

b) $a\sqrt{bx+c} - d\sqrt{ex+f} = 0 \quad | + d\sqrt{ex+f}$

$a\sqrt{bx+c} = d\sqrt{ex+f} \quad | ()^2$

$a^2(bx+c) = d^2(ex+f) \quad |$ ~~ausklammern~~ $\cancel{d}$ ausklammern

$a^2bx + a^2c = d^2ex + d^2f$

$\cancel{a^2(bx+c) = d^2(ex+f)}$

$x(a^2b) + a^2c = x(d^2e) + d^2f \quad | -d^2f$

$x(a^2b) + a^2c - d^2f = x(d^2e) \quad | -x(d^2e)$

$x(a^2b) + a^2c - d^2f - x(d^2e) = 0 \quad \cancel{| : x}$

$\cancel{(a^2b) - (d^2e) + \frac{a^2c - d^2f}{x} = 0}$

$x(a^2b - d^2e) + a^2c - d^2f = 0 \quad | : (a^2b - d^2e)$

$\boxed{x + \frac{a^2c - d^2f}{a^2b - d^2e} = 0}$

c) $\sqrt{x+5} - \sqrt{2x+3} = 1 \quad | + \sqrt{2x+3}$

$\sqrt{x+5} = 1 + \sqrt{2x+3} \quad | ()^2$

$x + 5 = \underbrace{1 + 2\cdot\sqrt{2x+3} + 2x + 3}_{\text{binomische Formel}} \quad | -3-1-2x$

$-x + 1 = 2\cdot\sqrt{2x+3} \quad | ()^2$

$x^2 - 2x + 1 = 4\cdot(2x+3) = 8x + 12 \quad | -8x-12$

$\underline{x^2 \cancel{+6} - 10x - 11 = 0}$

d) $\sqrt{x+2+\sqrt{2x+7}} = 4 \qquad | \; (\,)^2$

$x+2+\sqrt{2x+7} = 16 \qquad | \; -2-x$

$\sqrt{2x+7} = 14-x \qquad | \; (\,)^2$

$2x+7 = 196-28x+x^2 \qquad | \; -2x-7$

$\underline{\underline{x^2-30x+189=0}}$

e) $\dfrac{\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-1}} = \dfrac{1}{\sqrt{2x+10}} \qquad | \; \cancel{(\,)^2} \cdot \sqrt{2x+10}$

Auf gleichen Nenner bringen

$\cancel{\dfrac{x+3}{(x+3)+(\sqrt{x+3}\cdot\sqrt{x-1}\cdot(-2))}}$

$\cancel{x-3}$

$\sqrt{(x-3)(2x+10)} = \sqrt{x+3}-\sqrt{x-1} \qquad | \; (\,)^2$

$(x-3)(2x+10) = (x+3) - 2\sqrt{(x+3)(x-1)} + (x-1) \qquad |$ Ausklammern

$2x^2+10x-6x-30 = x+3+x-1 - 2\sqrt{x^2-x+3x-3} \qquad | \; -2x-2$

$2x^2+2x-32 = -2\sqrt{x^2+2x-3} \qquad | \; (\,)^2$

$4x^4+4x^2+1024+8x^3-128x^2-128x = 4(x^2+2x-3)$

$4x^4+8x^3-124x^2-128x+1024 = 4x^2+8x-12 \qquad | \; :4$

$x^4+2x^3-31x^2-32x+256 = x^2+2x-3 \qquad | \; -x^2-2x+3$

$\underline{\underline{x^4+2x^3-32x^2-34x+259=0}}$ ✓

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(1)

f) $\frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{3x+1}}=\sqrt{6x-1} \quad | \cdot \sqrt{3x+1}$

$\sqrt{x+4}-\sqrt{x}=\sqrt{(6x-1)(3x+1)} \quad | ()^2$

$(x+4)-2\cdot\sqrt{(x+4)x}+x=(6x-1)(3x+1)$

$2x+4-2\sqrt{x^2+4x}=18x^2+6x-3x-1 \quad | -2x-4$

$-2\sqrt{x^2+4x}=18x^2+x-5 \quad | ()^2$

$4(x^2+4x)=324x^4+x^2+25+36x^3-180x^2-10x$

$4x^2+16x=324x^4+36x^3-179x^2-10x+25 \quad | -4x^2-16x$

$324x^4+36x^3-183x^2-26x+25=0$ ✓

(2)

a) $x^2+x-56=0$ bezüglich $\mathbb{N}$

$x_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+56}$

$x_1=7$; $x_2=-8$

$\mathbb{L}=\{7\}$ oder $\mathbb{L}=\{x \mid x\in\mathbb{N} \text{ und } x^2+x-56=0\}=\{7\}$

b) $\frac{7x-5}{10x-3}=\frac{5x-3}{6x+1}$ bezüglich $\mathbb{Q}$

$\Rightarrow (7x-5)(6x+1)=(5x-3)(10x-3)$

$\Rightarrow 42x^2+7x-30x-5=50x^2-15x-30x+9 \quad | -42x^2-7x+30x+5$

$8x^2 \cancel{-22x} \cancel{+52x} -22x+14=0 \quad | :8$

$\cancel{x^2-2\frac{3}{4}x+\frac{1}{2}=0}$

$\cancel{x_{1,2}=1{,}375\pm\sqrt{1{,}375}}$

$\cancel{x_{1,2}=\frac{11}{8}\pm\sqrt{\left(\frac{11}{8}\right)^2-\frac{1}{2}}} \Rightarrow x_1=\cancel{1{,}390625}\ 2{,}55$

$x_2=0{,}1957$

→

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$$8x^2-22x+14=0 \quad |:8$$

$$x^2-\frac{22}{8}x+\frac{14}{8}=0$$

$$x_{1,2}=\frac{11}{8}\pm\sqrt{\left(\frac{11}{8}\right)^2-\frac{14}{8}}$$

$$\underline{\underline{x_1=1\frac{3}{4}=\frac{7}{4}}}$$

$$\underline{\underline{x_2=1}}$$

$$\mathbb{L}=\left\{1,\frac{7}{4}\right\}$$

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c) $\dfrac{x^2+2x-3}{x-1}=0$ bezüglich $\mathbb{R}$ $\quad |\cdot(x-1)$

$$x^2+2x-3=0$$

$$x_{1,2}=-1\pm\sqrt{1+3}$$

$\boxed{x_1=-1+2=1}$; $x_2=-1-2=-3$

~~$\mathbb{L}=\{1,-3\}$~~ $\lightning$ (s. Ausgangsgleichung) $\Rightarrow$ $\mathbb{L}=\{3\}$

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d) $\sqrt{2x-1}+\sqrt{x-4}-\sqrt{2x-6}-\sqrt{x-1}=0$ bezüglich $\mathbb{R}$

$$\sqrt{2x-1}+\sqrt{x-4}=\sqrt{2x-6}+\sqrt{x-1} \quad |(\,)^2$$

$$2x-1+2\cdot\sqrt{(x-1)(x-4)}+x-4=2x-6+2\sqrt{(2x-6)(x-1)}+x-1$$

$$\Rightarrow \cancel{2x+x}\ 3x-5+2\cdot\sqrt{x^2-5x+5}=3x-7+2\cdot\sqrt{2x^2-8x+6} \quad |-3x+7$$

$$2+2\cdot\sqrt{x^2-5x+5}=2\cdot\sqrt{2x^2-8x+6} \quad |\cancel{(\,)^2}:\left(2\cdot\sqrt{x^2-5x+5}\right)$$

~~$4+8\cdot\sqrt{x^2-5x+5}$~~

$$\cancel{4}\ 2=\frac{\sqrt{2x^2-8x+6}}{\sqrt{x^2-5x+5}} \quad |(\,)^2$$

$$2=\frac{2x^2-8x+6}{x^2-5x+5} \quad |\cdot(x^2-5x+5)$$

$$2\cdot\cancel{x}^2-10x+10=2x^2-8x+6 \quad |-2x^2+8x-6$$

$$-2x+4=0 \quad |:(-2)$$

$$x-2=0 \Rightarrow \underline{\underline{x=2}}$$

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2

e)

$$6x^2+x-1=0 \quad \text{bezüglich } \mathbb{N}, \mathbb{Q}, \mathbb{R} \qquad | :6$$

$$x^2+\frac{1}{6}-\frac{1}{6}=0$$

$$x_{1,2}=-\frac{1}{12}\pm\sqrt{\frac{1}{144}+\frac{1}{6}}$$

$$x_1=\frac{1}{3}\ ;\ x_2=-\frac{1}{2}$$

$$\mathbb{L}_{\mathbb{N}}=\cancel{\{\}}\ ;\ \mathbb{L}_{\mathbb{Q}}=\left\{\frac{1}{3},-\frac{1}{2}\right\}\ ;\ \mathbb{L}_{\mathbb{R}}=\left\{-0{,}5,\frac{1}{3}\right\}$$

f) $\frac{1}{x^2}$ bezüglich $\mathbb{R}$

$$\mathbb{L}=\emptyset$$

g)

$$\frac{2x^2-14x+6}{x-2}=0 \quad \text{bezüglich } \mathbb{R} \qquad |\cdot(x-2)$$

$$2x^2-14x+6=0 \quad |:2$$

$$x^2-7x+3=0$$

$$x_{1,2}=3{,}5\pm\sqrt{(3{,}5)^2-3}$$

$$x_1=6{,}54 \checkmark \qquad x_2=0{,}4586 \checkmark$$

$$\mathbb{L}=\{6{,}54,\ 0{,}459\}$$

oder auch: $\mathbb{L}=\left\{\frac{7}{2}+\sqrt{\frac{37}{4}},\ \frac{7}{2}-\sqrt{\frac{37}{4}}\right\}$

Aufgabe 3 Errate eine Lösung und errechne die weiteren Lösungen:

a) $x^3 + 3x^2 - 3x - 1 \Rightarrow x_1 = 1$ (erraten)

Polynomdivision:

$$
\begin{array}{l}
(x^3 + 3x^2 - 3x - 1) : (x - 1) = \underline{\underline{x^2 + 4x + 1}} \\
\underline{-(x^3 - x^2)} \\
\quad 4x^2 - 3x \\
\quad \underline{-(4x^2 - 4x)} \\
\qquad x - 1
\end{array}
$$

$$x^2 + 4x + 1 = 0$$

$$x_{2,3} = -2 \pm \sqrt{4 - 1}$$

$$\underline{\underline{x_2 = -2 + \sqrt{3}}}$$

$$\underline{\underline{x_3 = -2 - \sqrt{3}}}$$

$$\boxed{x_1 = 1 \;;\; x_2 = -2 + \sqrt{3} \;;\; x_3 = -2 - \sqrt{3}}$$

b) $x^3 + 7x^2 - 21x + 6 = 0$ erraten: $\boxed{x_1 = 2}$

$$
\begin{array}{l}
(x^3 + 7x^2 - 21x + 6) : (x - 2) = \underline{\underline{x^2 + 9x - 3}} \\
\underline{-(x^3 - 2x^2)} \\
\quad 9x^2 - 21x \\
\quad \underline{-(9x^2 - 18x)} \\
\qquad -3x + 6 \\
\qquad \underline{-(-3x + 6)} \\
\qquad\qquad 0
\end{array}
$$

$$x^2 + 9x - 3 = 0$$

$$\underline{x_2 = -4{,}5 + \sqrt{\tfrac{81}{4} + 3}} \; \cancel{\approx 0{,}32}$$

$$\underline{\underline{x_3 = -4{,}5 - \sqrt{\tfrac{81}{4} + 3}}}$$

$$L_{\mathbb{N}} = \{2\} \;;\; L_{\mathbb{R}} = \left\{2,\; -\tfrac{9}{2} + \sqrt{\tfrac{81}{4} + 3},\; -\tfrac{9}{2} - \sqrt{\tfrac{81}{4} + 3}\right\}$$

Übungsaufgaben -> Gleichungen höheren Grades

Aufgabe 1 Bestimme die Lösungsmenge in $\mathbb{R}$:

a) $15x^3 + x^2 - 24x = -22x \quad | + 22x$

$15x^3 + x^2 - 2x = 0 \quad$ | Ausklammern

$x(15x^2 + x - 2) = 0$

$\underline{x_1 = 0}$

$15x^2 + x - 2 = 0 \quad | : 15$

$x^2 + \frac{1}{15}x - \frac{2}{15} = 0$

$x_{1,2} = -\frac{1}{30} \pm \sqrt{\frac{1}{900} + \frac{120}{900}}$

$x_1 = \frac{1}{3} \quad ; \quad x_2 = -0{,}4$ => Probe ergibt: $-0{,}4$ passt nicht

$\underline{\underline{L = \{0, \frac{1}{3}\}}}$

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b) $(3x^2 - 20x - 7)x^2 = 0$

$(3x^3 - 20x^2 - 7x)x = 0 \quad \Rightarrow \boxed{x_1 = 0}$

$3x^3 - 20x^2 - 7x = 0 \quad$ raten: $\boxed{x_2 = 7}$

$(3x^3 - 20x^2 - 7x) : (x - 7) = \underline{\underline{3x^2 + x}}$

$-(3x^3 - 21x^2)$

$x^2 - 7x$

$-(x^2 - 7x)$

0

$3x^2 + x = 0$

$x(3x + 1) \Rightarrow 3x + 1 = 0 \quad | -1$

$3x = -1 \quad | : 3$

$\underline{x = -\frac{1}{3}} \longrightarrow \underline{\underline{x_3 = -\frac{1}{3}}}$

$\underline{\underline{L = \{-\frac{1}{3}, 0, 7\}}}$

c) $x^3 - 2ix^2 + 3x = 0 \qquad (i = \sqrt{-1})$

$x(x^2 - 2ix + 3) = 0 \qquad \Rightarrow \boxed{x_1 = 0}$

$\cancel{(x^2 - 2ix + 3) : (x}$

$x^2 - 2ix + 3 = 0$

$x_{2,3} = i\cancel{x} \pm \sqrt{-1-3} = \sqrt{-1} \pm \sqrt{-4}$

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d) $x^3 + x^2 - 25x - 5 = 20 \quad | -20$

$x^3 + x^2 - 25x - 25 = 0$

$\cancel{x(x^2 + x)}$

$x(x^2 - 25) + (x^2 - 25) = 0 \qquad \Rightarrow$ erraten: $x_1 = 5$

$(x^3 + x^2 - 25x - 25) : (x - 5) = \underline{x^2 + 6x + 5}$

$-(x^3 - 5x^2)$

$6x^2 - 25x$

$-(6x^2 - 30x)$

$5x - 25$

$-(5x - 25)$

0

$x^2 + 6x + 5 = 0$

$x_1 = -3 + \sqrt{9-5} = -3 + \sqrt{4} = -3 + 2 = \underline{-1}$

$x_2 = -3 - \sqrt{4} = -3 - 2 = \underline{\underline{-5}}$

$\underline{\mathbb{L} = \{-5, -1, 5\}}$

$\boxed{2}$ Berechne die Nullstellen in $\mathbb{R}$:

a) $x^3 + bx^2 - a^2x - ba^2 = 0$ $\qquad x_1 = -b$ (durch Probieren)

$\cancel{x(bx - a^2 + x^2) - ba^2 = 0}$

Polynomdivision:

$$(x^3 + bx^2 - a^2x - ba^2) : (x + b) = \underline{\underline{x^2 - a^2}}$$
$$-(x^3 + bx^2)$$
$$-a^2x - a^2b$$
$$-(-a^2x - a^2b)$$
$$0$$

$x^2 - a^2 = 0 \qquad | + a^2$

$x^2 = a^2 \qquad | \sqrt{\ }$

$x_{2,3} = \pm a$

$\underline{\underline{\mathbb{L} = \{-b, -a, a\}}}$

b) $x^3 + 14x^2 + 7x - 78 = 0$ $\qquad \boxed{x_1 = 2}$ durch Probieren

Polynomdivision

$$(x^3 + 14x^2 + 7x - 78) : (x - 2) = \underline{\underline{x^2 + 16x + 39}}$$
$$-(x^3 - 2x^2)$$
$$16x^2 + 7x$$
$$-(16x^2 - 32x)$$
$$39x - 78$$
$$-(39x - 78)$$
$$0$$

$x^2 + 16x + 39 = 0$

$x_{2,3} = -8 \pm \sqrt{64 - 39} = -8 \pm \sqrt{25} = -8 \pm 5$

$x_2 = -3 \quad ; \quad x_3 = -13$

$\underline{\underline{\mathbb{L} = \{-13, -3, 2\}}}$

$\boxed{2}$

c)

$$x^3 + x^2 - 20x = -2(x^2+10) + 4x^2 \quad | -4x^2 + 2(x^2+10)$$

$$x^3 - 3x^2 - 20x + 2(x^2+10) = 0$$

$$x^3 - 3x^2 - 20x + 2x^2 + 20 = 0$$

$$x^3 - x^2 - 20x + 20 = 0 \Rightarrow \boxed{x_1 = 1}$$ durch Probieren

Polynomdivision:

$$(x^3 - x^2 - 20x + 20) : (x-1) = \underline{x^2 - 20}$$
$$\frac{-(x^3 - x^2)}{\quad}$$
$$-20x + 20$$
$$\frac{-(-20x+20)}{0}$$

$$x^2 - 20 = 0 \Rightarrow x^2 = 20 \quad | \sqrt{}$$

$$\underline{\underline{x_{2,3} = \pm\sqrt{20}}}$$

$$\underline{\underline{\mathbb{L} = \{-\sqrt{20}, 1, +\sqrt{20}\}}}$$ oder $\mathbb{L} = \{-2\cdot\sqrt{5}, 1, +2\sqrt{5}\}$